Advanced Nonlinear Studies **12** (2012), ...

Liouville Type Theorems for Stable Solutions of Certain Elliptic Systems

Mostafa Fazly

Department of Mathematics
University of British Columbia, Vancouver, B.C. Canada V6T 1Z2
e-mail: fazly@math.ubc.ca

Received 10 December 2010
Communicated by Nassif Ghoussoub

Abstract

We establish Liouville type theorems for elliptic systems with various classes of non-linearities on $\mathbb{R}^N$. We show, among other things, that a system has no semi-stable solution in any dimension, whenever the infimum of the derivatives of the corresponding non-linearities is positive. We give some immediate applications to various standard systems, such as the Gelfand, and certain Hamiltonian systems. The case where the infimum is zero is more interesting and quite challenging. We show that any $C^2(\mathbb{R}^N)$ positive entire semi-stable solution of the following Lane-Emden system,

$$(N_{\lambda,\gamma}) \qquad \left\{ \begin{array}{rcl} -\Delta u &=& \lambda f(x)\, v^p, \\ -\Delta v &=& \gamma f(x)\, u^q, \end{array} \right.$$

is necessarily constant, whenever the dimension $N < 8 + 3\alpha + \frac{8+4\alpha}{q-1}$, provided $p = 1$, $q \geq 2$ and $f(x) = (1 + |x|^2)^{\frac{\alpha}{2}}$. The same also holds for $p = q \geq 2$ provided

$$N < 2 + \frac{2(2+\alpha)}{p-1}(p + \sqrt{p(p-1)}).$$

We also consider the case of bounded domains $\Omega \subset \mathbb{R}^N$, where we extend results of Brown et al. [1] and Tertikas [17] about stable solutions of equations to systems. At the end, we prove a Pohozaev type theorem for certain weighted elliptic systems.

2010 Mathematics Subject Classification. 35B08, 35B53, 35J61.
Key words. Liouville Type Theorems, Elliptic Systems, Entire Semi-Stable Solutions, Point-wise Estimates.

1 Introduction

In this paper, we shall examine semi-stable solutions of semi-linear elliptic systems on bounded and unbounded domains on $\mathbb{R}^N$. We are particularly interested in Liouville type theorems for stable solutions of such systems. Note that this subject is by now well developed for equations such as those involving Gelfand, Lane-Emden, MEMS and even more general C^1 convex non-linearities. To our knowledge, no such (Liouville type) theorem exists for stable solutions of systems corresponding to these non-linearities. Let us first mention the definition of stability.

Let Ω be a subset of $\mathbb{R}^N$ and $f, g \in C^1(\mathbb{R}^2, \Omega)$. Consider the following general system on Ω,

$$(N_{f,g}) \qquad \begin{cases} -\Delta u &= f(u,v,x), \\ -\Delta v &= g(u,v,x). \end{cases}$$

Following Montenegro [11] –whose work was restricted to bounded domains– we say that a solution $(u, v) \in C^2(\Omega)$ of $(N_{f,g})$ is *stable* if the following eigenvalue problem, $(S_{f,g})$, has a first positive eigenvalue, $\eta > 0$, with corresponding positive smooth eigenfunction pair (ϕ, ψ). We say it is *semi-stable* if the first eigenvalue η is non-negative, i.e., $\eta \geq 0$.

$$(S_{f,g}) \qquad \begin{cases} -\Delta \phi &= f_u(u,v,x)\phi + f_v(u,v,x)\psi + \eta\phi, \\ -\Delta v &= g_u(u,v,x)\phi + g_v(u,v,x)\psi + \eta\psi, \end{cases}$$

In section 2, we shall first consider systems on unbounded domains, and establish Liouville type results for the easy case where the infimum of the derivatives of the non-linearities are greater than zero. We then tackle the more challenging case when the infimum is zero so as to cover Lane-Emden systems.

In section 3, we give a general class of non-linearities for which Liouville type theorems can be established.

2 Systems on unbounded domains

Assume Ω is the whole space $\mathbb{R}^N$. We first prove the following result.

Theorem 2.1 *Assume f and g are two C^1-functions on $\mathbb{R}^{N+2}$.*

1. *If all partial derivatives of f and g with respect to u and v are non-negative, then there is no $C^2(\mathbb{R}^N)$ entire stable solution for $(N_{f,g})$, i.e., η must be zero.*

2. *If $\min\{\inf f_v, \inf g_u\} \geq C > 0$, where the infimum is on the range of u and v, and $f_u, g_v \geq 0$, then there is no $C^2(\mathbb{R}^N)$ entire semi-stable solution for $(N_{f,g})$.*

Proof. Since proofs of 1 and 2 are similar, we just prove 2. Suppose (u, v) is an entire semi-stable solution of $(N_{f,g})$, so that there exists a pair of positive functions (ϕ, ψ) satisfying $(S_{f,g})$ with $\eta \geq 0$. By adding two equations in $(S_{f,g})$ we get,

$$-\Delta(\phi + \psi) = (f_u + g_u)\phi + (f_v + g_v)\psi + \eta(\phi + \psi) \geq \inf g_u\, \phi + \inf f_v\, \psi,$$

therefore

$$\begin{aligned}\frac{-\Delta(\phi+\psi)}{\phi+\psi} &\geq (\inf g_u)\frac{\phi}{\phi+\psi}+(\inf f_v)\frac{\psi}{\phi+\psi}\\ &\geq \min\{\inf f_v, \inf g_u\}\geq C>0.\end{aligned}$$

Now, multiply the above equation by ζ^2 for $0\leq\zeta(x)\in C_c^1(\mathbb{R}^N)$, and do integration by parts to get

$$\begin{aligned}C\int_{\mathbb{R}^N}\zeta^2 &\leq \int_{\mathbb{R}^N}\frac{-\Delta(\phi+\psi)}{\phi+\psi}\zeta^2\\ &= \int_{\mathbb{R}^N}\frac{2\zeta}{\phi+\psi}\nabla(\phi+\psi)\cdot\nabla\zeta-\frac{|\nabla(\phi+\psi)|^2}{|\phi+\psi|^2}\zeta^2.\end{aligned}$$

Since the function $f(x):=2ax-x^2$ takes it maximum at $x=a$, we conclude

$$C\int_{\mathbb{R}^N}\zeta^2\leq\int_{\mathbb{R}^N}|\nabla\zeta|^2.$$

This is obviously a contradiction, since there is no Poincaré inequality on $\mathbb{R}^N$.

■

Examples: Let $\lambda,\gamma>0$. Consider the following systems on $\mathbb{R}^N$.

$$(N_{1,\lambda,\gamma})\qquad \begin{cases}-\Delta u = \lambda e^v,\\ -\Delta v = \gamma e^u,\end{cases}\ \text{Gelfand System}$$

$$(N_{2,\lambda,\gamma})\qquad \begin{cases}-\Delta u = \lambda(1+a(x)u+b(x)v)^p,\\ -\Delta v = \gamma(1+c(x)u+d(x)v)^q,\end{cases}\ \text{General Lane-Emden System}$$

$$(N_{3,\lambda,\gamma})\qquad \begin{cases}\Delta u = \lambda v^{-p},\\ \Delta v = \gamma u^{-q},\end{cases}\ \text{General MEMS System}$$

$$(N_{4,\lambda})\qquad \begin{cases}-\Delta u = v,\\ -\Delta v = \lambda f(u),\end{cases}\ \text{Bi-harmonic equation}$$

$$(N_{5,H})\qquad \begin{cases}-\Delta u = H_v(u,v),\\ -\Delta v = H_u(u,v),\end{cases}\ \text{Hamiltonian System}$$

where all $C^1(\mathbb{R}^N)$ functions a,b,c,d,p,q,f,H,H_u and H_v are positive.

Corollary 2.1 *There is no pair (u,v) of $C^2(\mathbb{R}^N)$ functions satisfying one of the following 4 conditions:*

1. *(u,v) is a positive entire semi-stable solution of either $(N_{1,\lambda,\gamma})$ or $(N_{2,\lambda,\gamma})$.*
2. *(u,v) is a positive bounded entire semi-stable solution of $(N_{3,\lambda,\gamma})$.*
3. *(u,v) is an entire semi-stable solution of the bi-harmonic equation $(N_{4,\lambda})$ with $f'\geq C>0$ in the range of u,*
4. *(u,v) is an entire semi-stable solution of the Hamiltonian system $(N_{5,H})$, where $H_{uv}\geq 0$ and $H_{uu},H_{vv}\geq C>0$ in the range of u and v.*

In the next part, we see how much the condition $\min\{\inf f_v, \inf g_u\} \geq C > 0$ in Theorem 2.1 on non-linearities is crucial. Without it, such as the case of Lane-Emden systems, the proof becomes much more delicate and will require restrictions on the dimension N.

2.1 The Lane-Emden system

Existence and non-existence of positive solutions for both Lane-Emden equations and systems have been interesting and challenging questions for decades.

Recently, Souplet [15] established the Lane-Emden conjecture in $N = 4$ and obtained a new region of non-existence for $N \geq 5$. The so-called Lane-Emden conjecture which has been open for at least fifteen years asserts that the elliptic system

$$\begin{cases} -\Delta u &= v^p \text{ in } \mathbb{R}^N, \\ -\Delta v &= u^q \text{ in } \mathbb{R}^N, \end{cases}$$

for $p, q > 0$ has no positive classical solution if and only if the pair (p, q) lies below the *Sobolev critical hyperbola*, i.e.

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}.$$

This statement is the analogue of the celebrated Gidas-Spruck [8] Liouville type theorem for the scalar case. Among other partial results, the conjecture had been proved for $N \leq 3$ in [16, 13] and for only radial solutions by Mitidieri in [9].

Also, the question of the regularity of semi-stable solutions of the system $-\Delta u = \lambda e^v$, $-\Delta v = \gamma e^u$ in Ω, and those of the fourth order equation $\Delta^2 u = f(u)$ with zero Dirichlet boundary conditions were examined in [2] and [3], respectively. Roughly speaking, there is a correspondence between the regularity of semi stable solutions on bounded domains and the non-existence of semi stable solutions on $\mathbb{R}^N$, via rescaling and a blow up procedure. We combine the techniques of the above mentioned papers to find Liouville type theorems in the notion of stability for a special case of Lane-Emden system in higher dimensions. During preparation of this work, we noticed that Wei-Ye [18] also used these techniques to establish independently Liouville type theorems for fourth order equations. To our knowledge, no Liouville type theorem have been established for entire semi-stable solutions of systems of the form

$$(N_{\lambda,\gamma}) \qquad \begin{cases} -\Delta u &= \lambda f(x)\, v^p, \\ -\Delta v &= \gamma f(x)\, u^q, \end{cases}$$

where $p \geq 1$, $q > 1$ and $\lambda, \gamma \in \mathbb{R}^+$.

Notation 1 Throughout this subsection, for the sake of simplicity, we say $f \preceq g$ if there exists a positive constant C such that $f \leq Cg$ holds.

We shall first need the following L^1-estimates, which were inspired by the work of Serrin and Zou [16] (whose proof is based on ODE techniques), and Mitidieri and Pohozaev [10], who prove similar results by a simpler PDE approach. See also [14].

Lemma 2.1 *Let $p \geq 1, q > 1$ and $\lambda, \gamma > 0$. For any $C^2(\mathbb{R}^N)$ positive entire solution (u, v) of $(N_{\lambda,\gamma})$ and $R > 1$, there holds*

$$\begin{aligned}\int_{B_R} f(x)v^p &\leq C_{\lambda,\gamma}R^{N-\frac{2(q+1)p}{pq-1}-\frac{p+1}{pq-1}\alpha},\\ \int_{B_R} f(x)u^q &\leq C_{\lambda,\gamma}R^{N-\frac{2(p+1)q}{pq-1}-\frac{q+1}{pq-1}\alpha},\end{aligned}$$

where $f(x) = (1+|x|^2)^{\frac{\alpha}{2}}$ for any $\alpha \in \mathbb{R}$, and $C_{\lambda,\gamma}$ does not depend on R.

Proof. Fix the following function $\zeta_R \in C_c^2(\mathbb{R}^N)$ with $0 \leq \zeta_R \leq 1$;

$$\zeta_R(x) = \begin{cases} 1, & \text{if } |x| < R;\\ 0, & \text{if } |x| > 2R;\end{cases}$$

where $\|\nabla\zeta_R\|_\infty \leq \frac{1}{R}$ and $\|\Delta\zeta_R\|_\infty \leq \frac{1}{R^2}$. For fixed $m \geq 2$, we have

$$|\Delta\zeta_R^m(x)| \leq \begin{cases} 0, & \text{if } |x| < R \text{ or } |x| > 2R;\\ R^{-2}\zeta_R^{m-2}, & \text{if } R < |x| < 2R.\end{cases}$$

For $m \geq 2$, test the first equation of $(N_{\lambda,\gamma})$ by ζ_R^m and integrate to get

$$\begin{aligned}\lambda\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}v^p\zeta_R^m &= -\int_{\mathbb{R}^N}\Delta u\zeta_R^m\\ &= -\int_{\mathbb{R}^N}u\Delta\zeta_R^m \leq R^{-2}\int_{B_{2R}\setminus B_R}u\zeta_R^{m-2}.\end{aligned}$$

Applying Hölder's inequality we get

$$\begin{aligned}\lambda\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}v^p\zeta_R^m &\leq R^{-2}\left(\int_{B_{2R}\setminus B_R}(1+|x|^2)^{\frac{-\alpha}{2q}q'}\right)^{\frac{1}{q'}}\\ &\quad\left(\int_{B_{2R}\setminus B_R}(1+|x|^2)^{\frac{\alpha}{2}}u^q\zeta_R^{(m-2)q}\right)^{1/q}\\ &\leq R^{(N-\frac{\alpha}{q}q')\frac{1}{q'}-2}\\ &\quad\left(\int_{B_{2R}\setminus B_R}(1+|x|^2)^{\frac{\alpha}{2}}u^q\zeta_R^{(m-2)q}\right)^{1/q}.\end{aligned}$$

By a similar calculation for $k \geq 2$, we obtain

$$\gamma\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}u^q\zeta_R^k \leq R^{-2}\int_{B_{2R}\setminus B_R}v\zeta_R^{k-2}(1+|x|^2)^{\frac{\alpha}{2p}}(1+|x|^2)^{\frac{-\alpha}{2p}}.$$

By Hölder's inequality we get

$$\gamma\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}u^q\zeta_R^k \leq R^{(N-\frac{\alpha}{p}p')\frac{1}{p'}-2}\left(\int_{B_{2R}\setminus B_R}(1+|x|^2)^{\frac{\alpha}{2}}v^p\zeta_R^{(k-2)p}\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{p'}=1$. For $p=1$ take $p'=\infty$. For large enough k we have $2+\frac{k}{q}<(k-2)p$, so we can choose m such that $2+\frac{k}{q}\le m\le(k-2)p$ which says $m\le(k-2)p$ and $k\le(m-2)q$. Therefore, by collecting above inequalities we get for $p\ge 1$,

$$\begin{aligned}
\gamma\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}u^q\zeta_R^k &\le R^{(N-\frac{\alpha}{p}p')\frac{1}{p'}-2}R^{\left((N-\frac{\alpha}{q}q')\frac{1}{q'}-2\right)\frac{1}{p}}\\
&\quad\left(\int_{B_{2R}\setminus B_R}(1+|x|^2)^{\frac{\alpha}{2}}u^q\zeta_R^k\right)^{1/pq},\\
\lambda\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}v^p\zeta_R^m &\le R^{\left((N-\frac{\alpha}{p}p')\frac{1}{p'}-2\right)\frac{1}{q}}R^{(N-\frac{\alpha}{q}q')\frac{1}{q'}-2}\\
&\quad\left(\int_{B_{2R}\setminus B_R}(1+|x|^2)^{\frac{\alpha}{2}}v^p\zeta_R^m\right)^{1/pq}.
\end{aligned}$$

Therefore, for $p\ge 1$ we have

$$\begin{aligned}
\int_{B_R}(1+|x|^2)^{\frac{\alpha}{2}}v^p &\le R^{N-\frac{2(q+1)p}{pq-1}-\frac{p+1}{pq-1}\alpha},\\
\int_{B_R}(1+|x|^2)^{\frac{\alpha}{2}}u^q &\le R^{N-\frac{2(p+1)q}{pq-1}-\frac{q+1}{pq-1}\alpha}.
\end{aligned}$$

■

Corollary 2.2 *With the same assumptions as Lemma 2.1, we have*

$$\int_{B_R}\frac{u^2}{(1+|x|^2)^{\frac{\alpha}{2}}}\le CR^{N-\frac{4(p+1)}{pq-1}-\frac{2(p+1)}{pq-1}\alpha-\alpha},$$

where the positive constant C does not depend on R.

Proof. Apply Hölder's inequality to obtain

$$\begin{aligned}
\int_{B_R}\frac{u^2}{(1+|x|^2)^{\frac{\alpha}{2}}} &\le \left(\int_{B_R}(1+|x|^2)^{\frac{\alpha}{2}}u^q\right)^{\frac{2}{q}}\left(\int_{B_R}(1+|x|^2)^{(-\frac{\alpha}{2}-\frac{\alpha}{q})\frac{q}{q-2}}\right)^{1-\frac{2}{q}}\\
&\le R^{N(1-\frac{2}{q})-(1+\frac{2}{q})\alpha}\left(\int_{B_R}(1+|x|^2)^{\frac{\alpha}{2}}u^q\right)^{\frac{2}{q}}.
\end{aligned}$$

Now, use Lemma 2.1 to get the desired inequality.

■

An immediate application of Lemma 2.1 is the following Liouville type theorem, in the absence of stability, for $(N_{\lambda,\gamma})$ which is related to the Lane-Emden conjecture. In the case $\alpha=0$, this was obtained by Mitidieri in [9] and also modified by Serrin and Zou in [16].

Theorem 2.2 *Let $p\ge 1, q>1$ and $\lambda,\gamma>0$. Assume (u,v) is a non-negative entire classical solution for $(N_{\lambda,\gamma})$, provided*

$$N-2\le\max\left\{\frac{(2+\alpha)(q+1)}{pq-1},\frac{(2+\alpha)(p+1)}{pq-1}\right\}.$$

Then, (u,v) must be the trivial solution.

Proof. Proof is a direct consequnce of Lemma 2.1. ∎

Now, we prove a point-wise comparison result for solutions of $(N_{\lambda,\gamma})$. The following lemma is an adaptation of a result of Souplet [15].

Lemma 2.2 *If (u, v) is a $C^2(\mathbb{R}^N)$ positive entire solution of $(N_{\lambda,\gamma})$, then we have*

$$u^{q+1} \leq \frac{q+1}{p+1} v^{p+1},$$

where $q \geq \max\{p, 2\}$, $\lambda \leq \gamma$ and $\alpha > -2$.

Proof. Define $w := u - \beta v^t$ for $0 < t \leq 1$ and $\beta > 0$, so we have

$$\begin{array}{rcl} \Delta w & = & \Delta u - \beta t \Delta v \, v^{t-1} - \beta t(t-1)|\nabla v|^2 v^{t-2} \\ & \geq & -\lambda(1+|x|^2)^{\frac{\alpha}{2}} v^p + \beta \, t\gamma(1+|x|^2)^{\frac{\alpha}{2}} u^q v^{t-1} \\ & \geq & \gamma(1+|x|^2)^{\frac{\alpha}{2}}(-v^p + \beta^{q+1} t \, v^{tq+t-1}), \qquad \text{on } \{w \geq 0\}. \end{array}$$

By taking $t = \frac{p+1}{q+1}$ and $\beta \geq (\frac{q+1}{p+1})^{\frac{1}{q+1}}$, we have $\Delta w \geq 0$ on $\{w \geq 0\}$. Now, by Green's theorem we have

$$\int_{B_R} |\nabla w_+|^2 = -\int_{B_R} w_+ \Delta w + \int_{|x|=R} w_+(x) \partial_\nu w(x) \, dS(x) \tag{2.1}$$

$$\leq R^{N-1} \int_{|z|=1} w_+(Rz) w_r(Rz) dS(z) = \frac{1}{2} R^{N-1} g'(R), \tag{2.2}$$

where $g(R) = \int_{|z|=1} w_+^2(Rz) dS(z)$. Moreover, on the set $\{w \geq 0\}$ we have $\beta v^t \leq u$ and u and v are positive, so

$$\begin{array}{rcl} g(R) & = & \displaystyle\int_{\{|z|=1\},\{w\geq 0\}} (u(Rz) - \beta v^t(Rz))^2 \, dS(z) \\ & \leq & \displaystyle\int_{\{|z|=1\}} u^2(Rz) \, dS(z) \\ & \leq & \displaystyle\left(\int_{\{|z|=1\}} u^q(Rz) \, dS(z)\right)^{2/q}, \qquad \text{since} \quad q \geq 2. \end{array}$$

Also, by Lemma 2.1, we know that

$$\begin{array}{rcl} \displaystyle\int_0^R r^{N-1} \int_{\{|z|=1\}} (1+|rz|^2)^{\frac{\alpha}{2}} u^q(rz) \, dS(z) dr & = & \displaystyle\int_0^R r^{N-1}(1+r^2)^{\frac{\alpha}{2}} \\ & & \displaystyle\int_{\{|z|=1\}} u^q(rz) \, dS(z) dr \\ & \leq & C_{\lambda,\gamma} R^{N - \frac{2(p+1)q}{pq-1} - \frac{q+1}{pq-1}\alpha}. \end{array}$$

Therefore, for $\alpha > -2$, there is a sequence $R_i \to \infty$ such that

$$\int_{\{|z|=1\}} u^q(R_i z) \, dS(z) \to 0.$$

This means that $g(R_i)\to 0$ for $R_i\to\infty$. Since g is a positive function, there is a sequence $\hat R_i\to\infty$ such that $g'(\hat R_i)\le 0$. Hence, from (2.1) we see that $\int_{B_{\hat R_i}}|\nabla w_+|^2=0$, i.e. w_+ is constant. If $w_+=C>0$, then by continuity we conclude $w=C$ and by definition of w we see $u\ge C>0$ in $\mathbb{R}^N$ which is in contradiction with Lemma 1. Therefore, $w_+=0$ and $u\le\beta v^t$.

■

Corollary 2.3 *Let (u,v) be a $C^2(\mathbb{R}^N)$ entire positive solution of $(N_{\lambda,\gamma})$ with $p=q\ge 2$, then $u=(\frac{\lambda}{\gamma})^{\frac{1}{p+1}}v$.*

It follows that $(N_{\lambda,\gamma})$ with $p=q\ge 2$ reduces to the single equation which has been classified by Farina in [7] for $\alpha=0$. Similar ideas as in Farina's easily yield the following result.

Theorem 2.3 *If (u,v) is a $C^2(\mathbb{R}^N)$ non-negative entire semi-stable solution of $(N_{\lambda,\gamma})$ with $p=q\ge 2$, and*

$$N<2+\frac{2(2+\alpha)}{p-1}(p+\sqrt{p(p-1)}),$$

then, (u,v) is the trivial solution.

Similar results were also obtained by Esposito-Ghoussoub-Guo [5, 6] and by Esposito [4]. They can be applied to obtain the following analogue for systems.

Theorem 2.4 *Suppose (u,v) is a $C^2(\mathbb{R}^N)$ non-negative entire semi-stable solution of $(N_{\lambda,\gamma})$ with $p=1$ in dimensions*

$$N<8+3\alpha+\frac{8+4\alpha}{q-1}. \tag{2.3}$$

Then, (u,v) is the trivial solution.

First, a simple calculation leads us to the following identity.

Lemma 2.3 *For any $w\in C^2(\mathbb{R}^N)$ and $\zeta\in C^2_c(\mathbb{R}^N)$, we have*

$$\begin{aligned}\frac{\Delta w}{(1+|x|^2)^{\frac{\alpha}{2}}}\Delta(w\zeta^2)-\frac{|\Delta(w\zeta)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} &= -4\frac{|\nabla w\cdot\nabla\zeta|^2}{(1+|x|^2)^{\frac{\alpha}{2}}}-\frac{w^2}{(1+|x|^2)^{\frac{\alpha}{2}}}|\Delta\zeta|^2\\ &+ 2\frac{w\Delta w}{(1+|x|^2)^{\frac{\alpha}{2}}}|\nabla\zeta|^2-2\frac{\nabla w^2\cdot\nabla\zeta}{(1+|x|^2)^{\frac{\alpha}{2}}}\Delta\zeta.\end{aligned}$$

Proof. Let $\lambda=\gamma=1$. Assume (u,v) is a semi-stable positive solution. Inspired by the linearised equation

$$\begin{aligned}-\Delta\phi &= (1+|x|^2)^{\frac{\alpha}{2}}\psi,\\ -\Delta\psi &= q(1+|x|^2)^{\frac{\alpha}{2}}u^{q-1}\phi,\end{aligned}$$

we have the following stability inequality, for all $\zeta\in C^2_c(\mathbb{R}^N)$

$$q\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}u^{q-1}\zeta^2\le\int_{\mathbb{R}^N}\frac{|\Delta\zeta|^2}{(1+|x|^2)^{\frac{\alpha}{2}}}. \tag{2.4}$$

Step 1. Test the stability inequality on u. Set $\zeta = u\xi$ for $\xi \in C_c^2(\mathbb{R}^N)$, and test the stability inequality on ζ to get

$$\begin{aligned}(q-1)\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}u^{q-1}|u\xi|^2 &\leq \int_{\mathbb{R}^N}\frac{|\Delta(u\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} - (1+|x|^2)^{\frac{\alpha}{2}}u^q u\xi^2\\ &= \int_{\mathbb{R}^N}\frac{|\Delta(u\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} + \int_{\mathbb{R}^N}u\Delta v\xi^2\\ &= \int_{\mathbb{R}^N}\frac{|\Delta(u\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} + \int_{\mathbb{R}^N}v\Delta(u\xi^2)\\ &= \int_{\mathbb{R}^N}\frac{|\Delta(u\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} - \int_{\mathbb{R}^N}\frac{\Delta u}{(1+|x|^2)^{\frac{\alpha}{2}}}\Delta(u\xi^2).\end{aligned}$$

Now, using Lemma 2.3, we get

$$\begin{aligned}(q-1)\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}u^{q-1}|u\xi|^2 &\leq 4\int_{\mathbb{R}^N}\frac{|\nabla u|^2|\nabla\xi|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} + \int_{\mathbb{R}^N}\frac{u^2}{(1+|x|^2)^{\frac{\alpha}{2}}}|\Delta\xi|^2\\ &+ 2\int_{\mathbb{R}^N}uv|\nabla\xi|^2 + 2\int_{\mathbb{R}^N}u^2 div(\frac{\nabla\xi\Delta\xi}{(1+|x|^2)^{\frac{\alpha}{2}}}).\end{aligned}$$

By Green's theorem, we can modify the first term in the right hand side as

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|\nabla u|^2|\nabla\xi|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} &= \int_{\mathbb{R}^N}\frac{u(-\Delta u)}{(1+|x|^2)^{\frac{\alpha}{2}}}|\nabla\xi|^2\\ &+ \frac{1}{2}\int_{\mathbb{R}^N}u^2 div\left(\frac{\nabla|\nabla\xi|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} + \nabla(1+|x|^2)^{\frac{-\alpha}{2}}|\nabla\xi|^2\right),\end{aligned}$$

combine this equality with the last inequality to get

$$\begin{aligned}\int_{\mathbb{R}^N}(1+|x|^2)^{\frac{\alpha}{2}}u^{q-1}|u\xi|^2 &\leq \int_{\mathbb{R}^N}uv|\nabla\xi|^2\\ &+ \int_{\mathbb{R}^N}\frac{u^2}{(1+|x|^2)^{\frac{\alpha}{2}}}\left(|\Delta\xi|^2 + |\nabla\xi\cdot\nabla\Delta\xi| + |\Delta|\nabla\xi|^2|\right)\\ &+ \int_{\mathbb{R}^N}\frac{u^2}{(1+|x|^2)^{\frac{\alpha+1}{2}}}\left(|\nabla\xi||\Delta\xi| + |\nabla|\nabla\xi|^2|\right)\\ &+ \int_{\mathbb{R}^N}\frac{u^2}{(1+|x|^2)^{\frac{\alpha+2}{2}}}|\nabla\xi|^2\\ &=: \int_{\mathbb{R}^N}uv|\nabla\xi|^2 + I(u,\xi).\end{aligned}$$

From this and the stability inequality, (2.4), we conclude

$$\int_{\mathbb{R}^N}\frac{|\Delta(u\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} \leq \int_{\mathbb{R}^N}uv|\nabla\xi|^2 + I(u,\xi). \tag{2.5}$$

Using the following equality

$$\Delta(u\xi) = -(1+|x|^2)^{\frac{\alpha}{2}} v\xi + u\Delta\xi + 2\nabla u \cdot \nabla\xi,$$

and (2.5) we get

$$\int_{\mathbb{R}^N} (1+|x|^2)^{\frac{\alpha}{2}} v^2\xi^2 \leq \int_{\mathbb{R}^N} uv|\nabla\xi|^2 + I(u,\xi).$$

Now, set $\xi = \zeta_R^m \in C_c^2(\mathbb{R}^N)$ for $0 \leq \zeta_R \leq 1$, $m > 2$, and

$$\zeta_R(x) := \begin{cases} 1, & \text{if } |x| < R; \\ 0, & \text{if } |x| > 2R; \end{cases}$$

where $||\nabla\zeta_R||_\infty \leq \frac{1}{R}$ and $||\Delta\zeta_R||_\infty \leq \frac{1}{R^2}$. So, by Young's inequality we get

$$\begin{aligned} \int_{\mathbb{R}^N} (1+|x|^2)^{\frac{\alpha}{2}} v^2\zeta_R^{2m} &\leq \int_{B_{2R}\setminus B_R} uv|\nabla\zeta_R|^2\zeta_R^{2(m-1)} + I(u,\zeta_R^m) \\ &\leq \frac{1}{2C}\int_{B_{2R}\setminus B_R} (1+|x|^2)^{\frac{\alpha}{2}} v^2\zeta_R^{2m} \\ &+ C\int_{B_{2R}\setminus B_R} \frac{u^2}{(1+|x|^2)^{\frac{\alpha}{2}}}|\nabla\zeta_R|^4\zeta_R^{2(m-2)} + I(u,\zeta_R^m). \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N} (1+|x|^2)^{\frac{\alpha}{2}} v^2\zeta_R^{2m} \leq R^{-4}\int_{B_{2R}\setminus B_R} \frac{u^2}{(1+|x|^2)^{\frac{\alpha}{2}}}\zeta_R^{2(m-2)} + I(u,\zeta_R^m). \tag{2.6}$$

On the other hand, by definition of $I(.,.)$ we have

$$\begin{aligned} I(u,\zeta_R^m) &\leq R^{-4}\int_{B_{2R}\setminus B_R} \frac{u^2}{(1+|x|^2)^{\frac{\alpha}{2}}}\zeta_R^{2(m-2)} \\ &+ R^{-3}\int_{B_{2R}\setminus B_R} \frac{u^2}{(1+|x|^2)^{\frac{\alpha+1}{2}}}\zeta_R^{2(m-2)} \\ &+ R^{-2}\int_{B_{2R}\setminus B_R} \frac{u^2}{(1+|x|^2)^{\frac{\alpha+2}{2}}}\zeta_R^{2(m-2)} \\ &\leq R^{-4}\int_{B_{2R}\setminus B_R} \frac{u^2}{(1+|x|^2)^{\frac{\alpha}{2}}}\zeta_R^{2(m-2)}. \end{aligned}$$

From this and (2.6), we get

$$\int_{\mathbb{R}^N} (1+|x|^2)^{\frac{\alpha}{2}} v^2\zeta_R^{2m} \leq R^{-4}\int_{B_{2R}\setminus B_R} \frac{u^2}{(1+|x|^2)^{\frac{\alpha}{2}}}\zeta_R^{2(m-2)},$$

in the light of Corollary 2.2, we see

$$\int_{\mathbb{R}^N} (1+|x|^2)^{\frac{\alpha}{2}} v^2\zeta_R^{2m} \leq R^{N-4-\alpha-\frac{8+4\alpha}{q-1}}. \tag{2.7}$$

Step 2. Test the stability inequality on v. Set $\zeta = v\xi$ for $\xi \in C_c^2(\mathbb{R}^N)$ and test the stability inequality, (2.4), on ζ to get

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^N} \frac{|\Delta(v\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} - q(1+|x|^2)^{\frac{\alpha}{2}} u^{q-1}|v\xi|^2 \\ &= \int_{\mathbb{R}^N} \frac{|\Delta(v\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} + qu^{q-1}\Delta u\, v\xi^2 \\ &= \int_{\mathbb{R}^N} \frac{|\Delta(v\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} + \Delta(u^q)v\xi^2 - q(q-1)|\nabla u|^2 u^{q-2} v\xi^2. \end{aligned}$$

Therefore,

$$\begin{aligned} q(q-1)\int_{\mathbb{R}^N} |\nabla u|^2 u^{q-2} v\xi^2 &\le \int_{\mathbb{R}^N} \frac{|\Delta(v\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} + \int_{\mathbb{R}^N} \Delta(u^q)v\xi^2 \\ &= \int_{\mathbb{R}^N} \frac{|\Delta(v\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} + \int_{\mathbb{R}^N} u^q \Delta(v\xi^2) \\ &= \int_{\mathbb{R}^N} \frac{|\Delta(v\xi)|^2}{(1+|x|^2)^{\frac{\alpha}{2}}} - \int_{\mathbb{R}^N} \frac{\Delta v}{(1+|x|^2)^{\frac{\alpha}{2}}} \Delta(v\xi^2). \end{aligned}$$

By the same idea as Step 1 and using Lemma 2.3 we get

$$\int_{\mathbb{R}^N} |\nabla u|^2 u^{q-2} v\xi^2 \le \int_{\mathbb{R}^N} u^q v|\nabla\xi|^2 + I(v,\xi). \tag{2.8}$$

On the other hand, by following the ideas of Cowan-Esposito-Ghoussoub [3], we have

$$\int_{\mathbb{R}^N} h(u)(-\Delta u)\xi^2 = \int_{\mathbb{R}^N} h'(u)|\nabla u|^2\xi^2 - \int_{\mathbb{R}^N} H(u)\Delta\xi^2,$$

where $H(u) = \int_0^u h(t)dt$. Let $h(u) := u^{\frac{3q-1}{2}}$, use Lemma 2.2 to get

$$\begin{aligned} \int_{\mathbb{R}^N} (1+|x|^2)^{\frac{\alpha}{2}} u^{2q}\xi^2 &\le \int_{\mathbb{R}^N} (1+|x|^2)^{\frac{\alpha}{2}} u^{\frac{3q-1}{2}} v\xi^2 \\ &\le \int_{\mathbb{R}^N} u^{\frac{3q-3}{2}} |\nabla u|^2\xi^2 + \int_{\mathbb{R}^N} u^{\frac{3q+1}{2}} |\Delta\xi^2| \\ &\le \int_{\mathbb{R}^N} u^{q-2} v|\nabla u|^2\xi^2 + \int_{\mathbb{R}^N} u^q v|\Delta\xi^2|. \end{aligned}$$

From this and (2.8) we see that

$$\int_{\mathbb{R}^N} (1+|x|^2)^{\frac{\alpha}{2}} u^{2q}\xi^2 \le \int_{\mathbb{R}^N} u^q v(|\nabla\xi|^2 + |\Delta\xi^2|) + I(v,\xi).$$

Using the same test function as Step 1 and doing similar calculation we get

$$\begin{aligned} \int_{\mathbb{R}^N} (1+|x|^2)^{\frac{\alpha}{2}} u^{2q}\zeta_R^{2m} &\le R^{-4}\int_{B_{2R}\setminus B_R} \frac{v^2}{(1+|x|^2)^{\frac{\alpha}{2}}} \zeta_R^{2(m-2)} \\ &\le R^{-4-2\alpha}\int_{B_{2R}} (1+|x|^2)^{\frac{\alpha}{2}} v^2. \end{aligned}$$

By the result of Step 1, (2.7), we get

$$\int_{B_{2R}} (1+|x|^2)^{\frac{\alpha}{2}} u^{2q} \quad \leq \quad R^{N-8-3\alpha-\frac{8+4\alpha}{q-1}}.$$

■

Remark One can see that Lemma 2.1 and 2.2 can be adopted for the following system. So, a counterpart of Theorem 2.4 can be proved.

$$(N'_{\lambda,\gamma}) \qquad \left\{ \begin{array}{rcl} -\Delta u &=& \lambda f(x)\, v, \\ -\Delta v &=& \gamma f(x)\, u^q + \gamma f(x)\, u^r. \end{array} \right.$$

Open problem: A natural question is how we can establish a Liouville type theorem for $(N_{\lambda,\gamma})$ with other values of p and q.

3 Systems on bounded domains

In this part, we generalize results of Brown et al. [1] and Tertikas [17] about the stable solutions of equations to systems. Consider the following system:

$$(N_{f,g}) \qquad \left\{ \begin{array}{rcll} -div(e^{\theta}\nabla u) &=& \lambda e^{\theta} f(v) & \text{in } \Omega, \\ -div(e^{\theta}\nabla v) &=& \gamma e^{\theta} g(u) & \text{in } \Omega, \end{array} \right.$$

with the Robin boundary conditions:

$$\left\{ \begin{array}{rcll} au + b\partial_\nu u &=& 0 & \text{on } \partial\Omega, \\ av + b\partial_\nu v &=& 0 & \text{on } \partial\Omega, \end{array} \right.$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $f, g \in C^2(\mathbb{R})$, $\theta \in C^1(\Omega)$, $a, b \in \mathbb{R}$ and $\lambda, \gamma \in \mathbb{R}^+$. In the special case $\theta = 0$, $\lambda = \gamma$ and $f = g$ we have the single equation.

We define $\mathbb{X}$ to be the following subset of convex functions in $C^2(\mathbb{R})$: $h \in \mathbb{X}$ if we have

$$\begin{array}{ll} \text{either} & h(0) < 0 \text{ and } h''(u) \geq 0 \text{ for } u > 0, \\ \text{or} & h(0) = 0 \text{ and } h''(u) > 0 \text{ for } u > 0. \end{array}$$

Similarly, concave function h belongs to $\mathbb{Y} \subset C^2(\mathbb{R})$ if we have

$$\begin{array}{ll} \text{either} & h(0) > 0 \text{ and } h''(u) \leq 0 \text{ for } u > 0, \\ \text{or} & h(0) = 0 \text{ and } h''(u) < 0 \text{ for } u > 0. \end{array}$$

For the following single equation with the Robin boundary condition,

$$(N_f) \qquad \left\{ \begin{array}{rl} -\Delta u = f(u) & \text{in } \Omega, \\ au + b\partial_\nu u = 0 & \text{on } \partial\Omega, \end{array} \right.$$

Brown and Shivaji in [1] proved the following result.

Theorem 3.1 *Suppose $f \in \mathbb{X}$ and $f'(u) > 0$ for $u > 0$, then all $C^2(\Omega)$ positive solutions of (N_f) are unstable.*

In the following theorem, Tertikas [17] improved the above result by removing the monotonicity condition $f'(u) > 0$, for $u > 0$.

Theorem 3.2 *(i) If $f \in \mathbb{X}$, then every $C^2(\Omega)$ positive solution of (N_f) is unstable.*
(ii) If $f \in \mathbb{Y}$, then all $C^2(\Omega)$ positive solutions of (N_f) are stable.

This theorem is sharp. Since for the Gelfand non-linearity $f(u) = e^u$ which we have $f(0) > 0$, by Montenegro's paper [11] we know that minimal solutions are stable. Now, we try to prove a counterpart of these theorems for systems.

Theorem 3.3 *(i) Let $f, g \in \mathbb{X}$, $\theta \in C^1(\Omega)$ and $\lambda, \gamma \in \mathbb{R}^+$. If (u, v) is a $C^2(\Omega)$ positive solution of $(N_{f,g})$, then (u, v) is unstable.*
(ii) Let $f, g \in \mathbb{Y}$, $\theta \in C^1(\Omega)$ and $\lambda, \gamma \in \mathbb{R}^+$. Then, all $C^2(\Omega)$ positive solutions of $(N_{\lambda,\gamma})$ are stable.

Proof. Let (u, v) be a positive semi-stable solution of $(N_{f,g})$, so there exists positive eigenfunction pair (ϕ, ψ) corresponding to the first eigenvalue η such that

$$(S_{f,g}) \qquad \begin{cases} -div(e^\theta \nabla\phi) &= \lambda e^\theta f'(v)\psi + \eta\phi \quad \text{in } \Omega, \\ -div(e^\theta \nabla\psi) &= \gamma e^\theta g'(u)\phi + \eta\psi \quad \text{in } \Omega, \end{cases}$$

with Robin boundary conditions:

$$\begin{cases} a\phi + b\partial_\nu\phi &= 0 \quad \text{on } \partial\Omega, \\ a\psi + b\partial_\nu\psi &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Multiply the first equation of $(S_{f,g})$ by v and the second equation of $(N_{f,g})$ by ϕ and integrate to get:

$$\begin{aligned} \lambda \int_\Omega e^\theta f'(v)v\psi + \int_\Omega \eta v\phi &= -\int_\Omega div(e^\theta\nabla\phi)v \\ &= -\int_\Omega div(e^\theta\nabla v)\phi + \int_{\partial\Omega} e^\theta\,(\phi\partial_\nu v - v\partial_\nu\phi) \\ &= \gamma\int_\Omega e^\theta g(u)\phi + \int_{\partial\Omega} e^\theta\,(\phi\partial_\nu v - v\partial_\nu\phi) \\ &= \gamma\int_\Omega e^\theta g(u)\phi, \end{aligned}$$

the last equality holds, since v and ϕ have the same boundary condition. Similarly,

$$\begin{aligned} \gamma \int_\Omega e^\theta g'(u)u\phi + \int_\Omega \eta u\psi &= -\int_\Omega div(e^\theta\nabla\psi)u \\ &= -\int_\Omega div(e^\theta\nabla u)\psi + \int_{\partial\Omega} e^\theta\,(\psi\partial_\nu u - u\partial_\nu\psi) \\ &= \lambda\int_\Omega e^\theta f(v)\psi + \int_{\partial\Omega} e^\theta\,(\psi\partial_\nu u - u\partial_\nu\psi) \\ &= \lambda\int_\Omega e^\theta f(v)\psi. \end{aligned}$$

For (i), since f is in $\mathbb{X}$, we have $f'(v)v > f(v)$. Therefore,

$$\begin{aligned}\gamma \int_\Omega e^\theta g(u)\phi &\geq \lambda \int_\Omega e^\theta f(v)\psi + \int_\Omega \eta v\phi \\ &= \gamma \int_\Omega e^\theta g'(u)u\phi + \eta \int_\Omega (u\psi + v\phi).\end{aligned}$$

Suppose (u, v) is a semi-stable solution, so $\eta \geq 0$ and we have

$$\gamma \int_\Omega e^\theta (g(u) - g'(u)u)\phi \geq 0, \tag{3.1}$$

and this is a contradiction, since $g \in \mathbb{X}$, i.e., $g(u) - g'(u)u < 0$.

For (ii), since f and g are in $\mathbb{Y}$, we have $f(v) > f'(v)v$ and $g(u) > g'(u)u$. Therefore,

$$\begin{aligned}\gamma \int_\Omega e^\theta g(u)\phi &\leq \lambda \int_\Omega e^\theta f(v)\psi + \int_\Omega \eta v\phi \\ &= \gamma \int_\Omega e^\theta g'(u)u\phi + \eta \int_\Omega (u\psi + v\phi),\end{aligned}$$

and,

$$\eta \int_\Omega (u\psi + v\phi) \geq \gamma \int_\Omega e^\theta (g(u) - g'(u)u)\phi > 0,$$

we conclude all eigenvalues are positive.

□

Remarks

1. There is no condition on f' and g'.
2. The same result holds for $f(u, v, x)$ and $g(u, v, x)$.
3. This theorem looks sharp. Since for the Gelfand non-linearity $f = g = \exp(.)$ which we have $f(0) > 0$ and $g(0) > 0$, by Montenegro's paper [11] we know that minimal solutions are stable. Moreover, he actually proved some properties of the curve $\Gamma = \partial\Lambda$, where

   $$\Lambda := \{(\lambda, \gamma) \in \mathbb{R}^+ : \text{there exists a smooth solution } (u, v) \text{ of } (N_{f,g})\}.$$

   We can easily get an upper bound for the curve $\Gamma \subseteq \{(\lambda, \gamma) \in \omega;\ \lambda\gamma \leq \frac{\lambda_1^2}{f'(0)g'(0)}\}$. Since $f'', g'' > 0$, we have $f(v) \geq f'(0)v$ and $g(u) \geq g'(0)u$. Multiply both equations with the first eigenfunction of the operator $-\Delta$, i.e. ϕ_1, and do integration by parts to get

   $$\lambda_1 \int_\Omega \phi_1 u \geq \lambda f'(0) \int_\Omega \phi_1 v \ \text{ and } \ \lambda_1 \int_\Omega \phi_1 v \geq \gamma g'(0) \int_\Omega \phi_1 u.$$

For the remainder of the paper, we will focus on Pohozaev type theorems for the following weighted elliptic system, in the absence of stability,

$$(N_{f,g}) \qquad \begin{cases} -\Delta u = |x|^\alpha f(v) & \text{in } \Omega, \\ -\Delta v = |x|^\beta g(u) & \text{in } \Omega, \end{cases}$$

where $f, g \in C^1(\mathbb{R})$ and $\alpha, \beta \geq 0$, with Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$.

Lemma 3.1 *All non-negative $C^2(\Omega)$ solutions of $(N_{f,g})$ satisfy*

$$\begin{aligned}\int_\Omega & ((N+\alpha)F(v)-(N-2)\tau\, vf(v))|x|^\alpha + \\ \int_\Omega & ((N+\beta)G(u)-(N-2)(1-\tau)\, ug(u))|x|^\beta = \\ \int_{\partial\Omega} & (x\cdot\nu)\partial_\nu u\partial_\nu v dS,\end{aligned}$$

for any τ in $[0,1]$. The functions F and G stand for anti-derivative functions of f and g, respectively, i.e., $F(t) := \int_0^t f(s)ds$.

Proof. First, we show the following identity holds for any $C^2(\Omega)$ functions u and v which are zero on the boundary of Ω.

$$\int_\Omega (x\cdot\nabla v)\Delta u + (x\cdot\nabla u)\Delta v - (N-2)\nabla u\cdot\nabla v = \int_{\partial\Omega}(x\cdot\nu)\partial_\nu u\partial_\nu v dS. \tag{3.2}$$

By a straightforward calculation, one can observe the following identities:

$$\begin{aligned} div((x\cdot\nabla v)\nabla u &+ (x\cdot\nabla u)\nabla v) \\ &= (x\cdot\nabla v)\Delta u + (x\cdot\nabla u)\Delta v + x\cdot\nabla(\nabla u\cdot\nabla v) + 2\nabla u\cdot\nabla v, \\ div(x(\nabla u\cdot\nabla v)) &= N\nabla u\cdot\nabla v + x\cdot\nabla(\nabla u\cdot\nabla v).\end{aligned}$$

Subtract these equalities to get

$$\begin{aligned} div((x\cdot\nabla v)\nabla u &+ (x\cdot\nabla u)\nabla v - x(\nabla u\cdot\nabla v)) \\ &= (x\cdot\nabla v)\Delta u + (x\cdot\nabla u)\Delta v - (N-2)\nabla u\cdot\nabla v.\end{aligned}$$

Integrating over Ω and using Green's formula, we get

$$\begin{aligned}\int_\Omega & ((x\cdot\nabla v)\Delta u + (x\cdot\nabla u)\Delta v - (N-2)\nabla u\cdot\nabla v)\, dx = \\ \int_{\partial\Omega} & ((x\cdot\nabla v)\nabla u + (x\cdot\nabla u)\nabla v - x(\nabla u\cdot\nabla v))\cdot\nu dS.\end{aligned}$$

Since $u=v=0$ on $\partial\Omega$, we have $\nabla u = \partial_\nu u\,\nu$ and $\nabla v = \partial_\nu u\,\nu$. This proves (3.2).

Now, assume (u,v) is a solution of $(N_{f,g})$. Then,

$$\begin{aligned}(x\cdot\nabla v)\Delta u + (x\cdot\nabla u)\Delta v &= -(x\cdot\nabla v)|x|^\alpha f(v) - (x\cdot\nabla u)|x|^\beta g(u) \\ &= -x\cdot\nabla F(v)|x|^\alpha - x\cdot\nabla G(u)|x|^\beta \\ &= -div\left(x|x|^\alpha F(v) + x|x|^\beta G(u)\right) \\ &+ (N+\alpha)|x|^\alpha F(v) + (N+\beta)|x|^\beta G(u).\end{aligned}$$

Therefore, we have

$$\int_\Omega ((x\cdot\nabla v)\Delta u + (x\cdot\nabla u)\Delta v)\, dx = \int_\Omega (N+\alpha)|x|^\alpha F(v) + (N+\beta)|x|^\beta G(u). \tag{3.3}$$

On the other hand, by multiplying equations of $(N_{f,g})$ with u and v, we get

$$\begin{aligned}\int_\Omega \nabla u\cdot\nabla v &= -\int_\Omega v\Delta u dx = \int_\Omega |x|^\alpha v f(v) dx, \\ \int_\Omega \nabla u\cdot\nabla v &= -\int_\Omega u\Delta v dx = \int_\Omega |x|^\beta u g(u) dx.\end{aligned}$$

Therefore, for any $\tau \in [0,1]$ we have

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega \Big(\tau |x|^\alpha v f(v) + (1-\tau)|x|^\beta u g(u)\Big)\, dx. \tag{3.4}$$

Substitute (3.3) and (3.4) into (3.2) to get the desired result.

■

Theorem 3.4 *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a star-shaped and bounded domain. Then, there is no $C^2(\Omega)$ positive solution for $(N_{f,g})$, provided f and g satisfy the following conditions for all $t > 0$:*

$$\begin{aligned}(N+\alpha)F(t) &\leq a_1 t f(t),\\ (N+\beta)G(t) &\leq a_2 t g(t),\end{aligned}$$

where $a_1, a_2 \in \mathbb{R}^+$ and $a_1 + a_2 = N-2$.

Proof. Since Ω is a star-shaped domain, we have $\int_{\partial\Omega}(x\cdot\nu)\partial_\nu u \partial_\nu v dS > 0$. Therefore, Lemma 3.1 leads us to a contradiction if we take $(N+\alpha)F(v) - (N-2)\tau\, v f(v) \leq 0$ and $(N+\beta)G(u) - (N-2)(1-\tau)\, u g(u) \leq 0$.

■

Therefore, for the weighted Lane-Emden system we have the following result.

Corollary 3.1 *Let $f(v) = v^p$ and $g(u) = u^q$ for $p, q \geq 1$ in Theorem 3.4. Then, there is no positive classical solution for $(N_{f,g})$ in dimensions*

$$\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} \leq N-2.$$

Remark For the fourth order equation, i.e. $\alpha = 0, p = 1$ and $N > 4$, there is no positive solution for $q \geq \frac{N+4+2\beta}{N-4}$ and also for the weighted Lane-Emden equation, i.e., $p = q$ and $\alpha = \beta$, there is no positive solution for $q \geq \frac{N+2+2\beta}{N-2}$. This equation is also called the Hénon equation, see [12].

Acknowledgement. I would like to thank Professor Nassif Ghoussoub, my supervisor, for his constant support. Also, many thanks to friend of mine Dr. Craig Cowan for his comments.

References

[1] K. J. Brown, R. Shivaji; *Instability of nonnegative solutions for a class of semipositone problems*, Proc. Amer. Math. Soc. **112** (1991), 121-124.

[2] C. Cowan; *Regularity of the extremal solutions in a Gelfand system problem*, Adv. Nonlinear Stud. **11** (2011), 695-700.

[3] C. Cowan, P. Esposito, N. Ghoussoub; *Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains*, Discrete and Continuous Dynamical Systems - Series A, **28** (2010), 1033-1050.

[4] P. Esposito, *Linear instability of entire solutions for a class of non-autonomous elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A, **138** (2008), 1005-1018.

[5] P. Esposito, N. Ghoussoub, Y. Guo; *Compactness along the branch of semistable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math. **60** (2007), 1731-1768.

[6] P. Esposito, N. Ghoussoub, Y. Guo; Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS, Courant Lecture Notes in Mathematics **20**, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2010, xiv+318 pp.

[7] A. Farina; *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$, J. Math. Pures Appl. (9) **87** (2007), 537-561.

[8] B. Gidas, J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Commun. Pure Appl. Math. **34** (1981), 525-598.

[9] E. Mitidieri, *Nonexistence of positive solutions of semilinear elliptic systems in* $\mathbb{R}^N$, Differential Integral Equations **9** (1996), 465-479.

[10] E. Mitidieri, S. I. Pokhozhaev; *A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities*, Tr. Mat. Inst. Steklova, (2001) 234:1-384.

[11] M. Montenegro; *Minimal solutions for a class of elliptic systems*, Bull. London Math. Soc. **37** (2005), 405-416.

[12] W. M. Ni, *A nonlinear Dirichlet problem on the unit ball and its applications*, Indiana Univ. Math. J. **31** (1982), 801-807.

[13] P. Polácik, P. Quittner, Ph. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems, Part I: elliptic systems*, Duke Math. J. **139** (2007), 555-579.

[14] P. Quittner, Ph. Souplet; Superlinear Parabolic Problems. Blow-Up, Global Existence and Steady States, Birkhauser Verlag, Basel, 2007.

[15] Ph. Souplet; *The proof of the Lane-Emden conjecture in four space dimensions*, Adv. Math. **221** (2009), 1409-1427.

[16] J. Serrin, H. Zou; *Non-existence of positive solutions of Lane-Emden systems*, Differential Integral Equations **9** (1996), 635-653.

[17] A. Tertikas; *Stability and Instability of Positive Solutions of Semipositone Problems*, Proc. Amer. Math. Soc. **114** (1992), 1035-1040.

[18] J. Wei, D. Ye; *Liouville theorems for finite Morse index solutions of biharmonic problem*, preprint.